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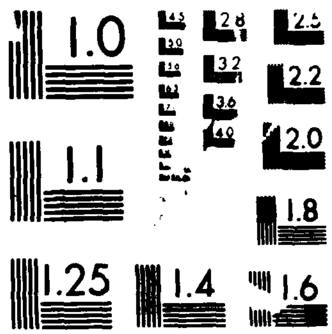


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# CENTER FOR STOCHASTIC PROCESSES

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PROBABILITY BOUNDS FOR M-SKOROHOD OSCILLATIONS

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# PROBABILITY BOUNDS FOR M-SKOROHOD OSCILLATIONS

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**Abstract**

*Billingsley developed a widely used method for proving weak convergence with respect to the sup-norm and* $J_1$*-Skorohod topologies, once convergence of the finite-dimensional distributions has been established. Here we show that Billingsley's method works not only for J oscillations, but also for M oscillations. This is done by identifying a common property of the J and M functions, called sub-triadditivity, and then showing that Billingsley's approach in the case of the J function can be adequately modified to apply to any sub-triadditive function.*



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## 1. Statement of results

Billingsley (1968) developed a widely used method for proving weak convergence with respect to the sup-norm and $J_1$-Skorohod topologies, once convergence of the finite-dimensional distributions has been established. The idea is to replace the evaluation of probabilities of large oscillations by the evaluation of the probability of large increments at fixed given times. Here we show that Billingsley's method can be made to work not only for $J$ oscillations, but also for $M$ oscillations. We also investigate a limiting case.

We use these results in Avram and Taqqu (1987) to study the weak convergence to the Lévy $\alpha$-stable process of normalized sums of moving averages that have at least two non-zero coefficients. In that paper, we show that $J_1$-weak convergence does not hold and we provide sufficient conditions for $M_1$ convergence.

Let

$$J(x_1, x_2, x_3) = \min(|x_2 - x_1|, |x_3 - x_2|). \tag{1.1a}$$

$$\begin{aligned} M(x_1, x_2, x_3) &= \text{the distance from } x_2 \text{ to } [x_1, x_3] \\ &= \begin{cases} 0 & \text{if } x_2 \in [x_1, x_3] \\ J(x_1, x_2, x_3) & \text{otherwise.} \end{cases} \end{aligned} \tag{1.1b}$$

Let $H$ stand for either $J$ or $M$, set

$$H_Z(t_1, t, t_2) = H(Z(t_1), Z(t), Z(t_2)).$$

and define the $H$ oscillation of $Z(t)$ as

$$\omega_\delta^H(Z) = \sup_{\substack{t_1 \le t \le t_2 \\ 0 < t_2 - t_1 < \delta}} H_Z(t_1, t, t_2). \tag{1.2}$$

This $\omega_\delta^H$ is of interest because, if $Z_n$ are $D[0,1]$ processes whose finite-dimensional distributions converge to those of a process $Z$, then the $H_1$-weak convergence of $Z_n$ to $Z$ is equivalent to the convergence in probability of $\omega_\delta^H(Z_n)$ to 0, uniformly in $n$, as $\delta$ approaches 0 (see Skorohod (1956), Theorem 3.2.1).

Throughout the paper, we will consider a process $Z(t)$, $0 \le t \le 1$, satisfying the assumption

$$(A) \qquad P\left\{H_Z(t_1,t,t_2) \ge \epsilon\right\} \le L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta},$$

for some constants $L > 0$, $\nu > 0$ and $\beta \ge 0$.

We shall also assume that

(B) There exists a number $n$ so that $Z(t)$ is pathwise constant on $(\frac{i}{n}, \frac{i+1}{n}]$ for $i = 0, 1, \ldots, n-1$.

Condition (B) is one of convenience and it will be weakened in the case $\beta > 0$ (see Corollary 1 below.)

The following result states that $P\{\omega_\delta^H(Z) \ge \epsilon\}$ satisfies a bound similar to that of $P\{H_Z(t_1,t,t_2) \ge \epsilon\}$ given in (A).

**Theorem 1.** *Let $H$ stand for either $J$ or $M$. Let $Z(t)$ be a process satisfying assumptions (A) and (B). Then, $Z$ also satisfies*

$$(1.3) \qquad P\{\omega_\delta^H(Z) \ge \epsilon\} \le C(\nu,\beta,n)L\epsilon^{-\nu}\delta^{\beta},$$

*where*

$$C(\nu,\beta,n) = \begin{cases} C(\nu,\beta) & \text{if } \beta > 0 \\ C(\nu,\beta)(\ln n)^{2\nu+2} & \text{if } \beta = 0 \end{cases}$$

*denotes a constant independent of $\epsilon, \delta$ and the distribution of $Z$.*

Theorem 1 is proved in this section.

**Remarks.** (1) Throughout the paper, we adopt the convention that "constants" may depend on $\nu$ and $\beta$, but not on $\epsilon, \delta$ and the distribution of $Z$. Dependence or independence on $n$ will be spelled out in each case.

(2) When $H = J$, Theorem 1 reduces basically to Theorem 12.5 of Billingsley (1968).

(3) When $\beta > 0$, $C(\nu,\beta,n)$ is independent of $n$. Assumption (B) can then be replaced by $Z(t) \in D[0,1]$ a.s.. Indeed, fix $n$ and divide $[0,1]$ in $2^n$ equal parts; then, apply Theorem 1 to the step function approximation $Z_n(t)$ of $Z(t)$ built by using the values of

$Z$ at $i/2^n, i = 1, \ldots, 2^n$. Since $Z(t) \in D[0,1]$ a.s. implies that almost all paths of $Z$ are right-continuous and $\omega_\delta^H(Z_n) \to \omega_\delta^H(Z)$ a.s., we have.

$$P\{\omega_\delta^H(Z) \geq \epsilon\} = \lim_{n \to \infty} P\{\omega_\delta^H(Z_n) \geq \epsilon\} \leq C(\nu, \beta) L \epsilon^{-\nu} \delta^\beta,$$

and hence

**Corollary 1.** *Let $H$ stand for either $J$ or $M$ and let $Z(t)$ be a process satisfying (A), with $\beta > 0$, and with paths in $D[0,1]$ a.s.. Then*

$$P\{\omega_\delta^H(Z) \geq \epsilon\} \leq C(\nu, \beta) L \epsilon^{-\nu} \delta^\beta, \tag{1.4}$$

*where $C(\nu, \beta)$ is a constant.*

Theorem 1 is proved by identifying a common property of the $J$ and $M$ functions, which we call *sub-triadditivity*, and then by showing that Billingsley's approach in the case of the $J$ function can be adequately modified to apply to any sub-triadditive function.

First some notation; let

$$[0,1]^3_\leq = \left\{ (t_1, t, t_2) : \ 0 \leq t_1 \leq t \leq t_2 \leq 1 \right\}$$

For any function $f : [0,1]^3_\leq \to \mathbf{R}^+$, and interval $I = [t_1, t_2]$, we introduce three new functions:

$$\overline{f}_I = \overline{f}(t_1, t_2) := \sup_{t \in [t_1, t_2]} f(t_1, t, t_2). \tag{1.5}$$

$f^* : [0,1]^3_\leq \to \mathbf{R}^+$, defined by

$$f^*(t_1, t, t_2) := \sup_{\substack{t_1' \in [t_1, t] \\ t_2' \in [t, t_2]}} f(t_1', t, t_2'), \tag{1.6}$$

and

$$\overline{f}^*_I = \overline{f}^*(t_1, t_2) := \sup_{t_1 \leq a \leq b \leq c \leq t_2} f(a, b, c). \tag{1.7}$$

**Billingsley's method consists of showing successively that the bound in assumption (A) leads to similar bounds for probabilities involving $\overline{H}_Z, \overline{H}^*_Z$ and $\omega_\delta^H(Z)$. The next theorem**

is our basic result. It identifies conditions for the first two steps of the extension, namely $H_Z \to \overline{H}_Z \to \overline{H}^*_Z$ and uses the notion of sub-triadditivity defined in Section 3.

**Theorem 2.** *(a) Let $Z(t)$ be a process satisfying assumptions (A) and (B), and let $f_Z$ be a random function: $[0,1]^3_{\le} \to \mathbf{R}^+$ which is a.s. inner sub-triadditive (see (3.1), (3.2) and (3.8) for a definition).*

*If*

(1.8)
$$P\left\{f_Z(t_1,t,t_2) \geq \epsilon\right\} \leq L\epsilon^{-\nu}(t_2-t_1)^{1+\beta}$$

*for some constants $L > 0$, $\nu > 0$ and $\beta \geq 0$, then*

(1.9)
$$P\left\{\overline{f}_Z(t_1,t_2) \geq \epsilon\right\} \leq 2K(\nu,\beta,n)L\epsilon^{-\nu}(t_2-t_1)^{1+\beta}$$

*where*

$$K(\nu,\beta,n) = \begin{cases} K(\nu,\beta) & \text{if } \beta > 0, \\ K(\nu,\beta)(\log_2 n)^{\nu+1} & \text{if } \beta = 0 \end{cases}$$

*(see (2.4) for the exact formula for the constant $K(\nu,\beta,n)$).*

*(b) If $f_Z$ is in addition outer sub-triadditive (see (3.3),(3.4) for a definition), then*

(1.10)
$$P\left\{\overline{f}^*_Z(t_1,t_2) \geq \epsilon\right\} \leq A(\nu,\beta,n)L\epsilon^{-\nu}(t_2-t_1)^{1+\beta}$$

*where*

$$A(\nu,\beta,n) = \begin{cases} A(\nu,\beta) & \text{if } \beta > 0 \\ A(\nu,\beta)(\log_2 n)^{2\nu+2} & \text{if } \beta = 0 \end{cases}$$

*(see (3.10) for the exact formula for the constant $A(\nu,\beta,n)$.)*

Theorem 2 is proved in Section 3.

The last step of the extension $\overline{H}^*_Z \to \omega^H_\delta(Z)$ always works, as the next Lemma shows.

**Lemma 1.** *If*

$$P\left\{\overline{f}^*_Z(t_1,t_2) \geq \epsilon\right\} \leq L\epsilon^{-\nu}(t_2-t_1)^{1+\beta}$$

*then*

$$P\left\{\omega_\delta(f_Z) \geq \epsilon\right\} \leq 2^{1+\beta}L\epsilon^{-\nu}\delta^\beta.$$

*where*

$$\omega_\delta(f_Z) := \sup_{\substack{t_1 \le t \le t_2 \\ 0 \le t_2 - t_1 \le \delta}} f_Z(t_1, t, t_2).$$

**Proof of Lemma 1.** Let $m = \lceil \delta^{-1} \rceil \equiv [\delta^{-1}] + 1$ and partition $[0, 1]$ with $t_i$, $i = 1, \ldots, m$, $t_{i+1} - t_i < \delta$ where $t_0 = 0$ and $t_{m-1} = 1$. Note that $\{\omega_\delta(f_Z) \ge \epsilon\}$ implies $\{\max_{i=0,\ldots,m-1} \overline{f}^*_Z(t_i, t_{i+2}) \ge \epsilon\}$. Hence

$$\begin{aligned} P\Big\{\omega_\delta(f_Z) \ge \epsilon\Big\} &\le \sum_{i=1}^{m-1} P\Big\{\overline{f}^*_Z(t_i, t_{i+2}) \ge \epsilon\Big\} \\ &\le (m-1) L\epsilon^{-\nu} (2\delta)^{1+\beta} \\ &\le 2^{1+\beta} L \epsilon^{-\nu} \delta^\beta. \end{aligned}$$

∎

Theorem 1 follows now as a particular case of Theorem 2(b), since the functions $J_Z(t_1, t, t_2)$ and $M_Z(t_1, t, t_2)$ are a.s. inner and outer sub-triadditive (see Appendix).

When $\beta = 0$, a better bound than in (1.3) may be obtained, if $Z$ satisfies a stronger assumption than (A), namely

$$(A') \qquad P\Big\{H^*_Z(t_1, t, t_2) > \epsilon\Big\} \le L\epsilon^{-\nu}(t_2 - t_1)^{1-\beta},$$

where

$$(1.11) \qquad H^*_Z(t_1, t, t_2) := \sup_{\substack{t_1' \in [t_1, t] \\ t_2' \in [t, t_2]}} H_Z(t_1', t, t_2'),$$

and where $H$ stands for either $J$ or $M$. Because the function $H^*_Z$ is inner sub-triadditive (see Appendix), one can apply Theorem 2(a). This yields

$$P\{\overline{H}^*_Z(t_1, t_2) \ge \epsilon\} \le 2[\log_2(4n)]^{\nu+1} L\epsilon^{\nu+1}$$

where we used (2.4) to evaluate $K(\nu, 0, n)$. Applying Lemma 1, we obtain

**Theorem 3.** *Let $H$ stand for either $J$ or $M$, and let $Z(t)$ satisfy assumption $(A')$ with $\beta = 0$, and assumption (B). Then*

$$(1.12) \qquad P\Big\{\omega^H_\delta(Z) \ge \epsilon\Big\} \le 4[\log_2 (4n)]^{\nu+1} L\epsilon^{-\nu}.$$

The paper is organized as follows:

In Section 2 we give a general formulation of the classical method of bisection used in Billingsley (1968).

In Section 3 we define sub-triadditivity and prove Theorem 2. using the method of bisection.

The sub-triadditivity of $J$, $M$, and $f_Z^*$ is established in the Appendix.

## 2. The Bisection Method

The bisection method seems to have originated with Menchoff (1923) in the context of a maximal inequality for partial sums of orthogonal random variables (see Stout (1974), Th. 2.3.1). It is used by Billingsley (1968) to establish a more general maximal inequality. The following proposition is a formalization of the bisection method.

Consider a random field $g_I$ indexed by subintervals I of $[0,1]$. For every $I \subset [0,1]$, denote by $I'$ the first half of $I$, and by $I''$ the second half of I. Let $m(\ )$ denote Lebesgue measure.

**Proposition 1**. *Suppose that a random field $g_I$ satisfies the inequality*

$$g_I \leq \max(g_{I'}, g_{I''}) + h_I \tag{2.1}$$

*where*

$$P\{h_I \geq \epsilon\} \leq M\epsilon^{-\nu}(m(I))^{1+\beta}, \tag{2.2}$$

*with $M$ constant, $\nu > 0$, $\beta \geq 0$. Suppose also that there exists an integer $n$ such that for every $J \subset [0,1]$ with $m(J) < \frac{1}{n}$, one has*

$$P\{g_J \geq \epsilon\} \leq M\epsilon^{-\nu}(m(J))^{1+\beta}. \tag{2.3}$$

*Then, for every $I \subset [0,1]$,*

$$P\{g_I \geq \epsilon\} \leq M\ K(\nu, \beta, n)\epsilon^{-\nu}(m(I))^{1+\beta}$$

*where*

$$K(\nu,\beta,n) = \begin{cases} [1-2^{-\beta/(\nu+1)}]^{-(\nu+1)} & \text{if } \beta > 0 \\ [\log_2(4n)]^{\nu+1} & \text{if } \beta = 0. \end{cases} \tag{2.4}$$

**Remark**. In general, we would expect $M' = MK(\nu,\beta,n) \approx nM$, since $[0,1]$ is composed of $n$ intervals on which the bound (2.3) holds. Proposition 1, however, shows that, in fact, when $\beta > 0$, $M'$ can be made independent of $n$, and even for $\beta = 0$ the growth is at most logarithmic in $n$.

**Proof**. We split $I \subset [0,1]$ in two halves, split each half again, and so on, until, after

$$k = \lceil \log_2 \lceil n \cdot m(I) \rceil \rceil \leq \log_2 \lceil nm(I) \rceil + 1 = \log_2 \, 2\lceil nm(I) \rceil \leq \log_2(2n)$$

splittings, we end up with intervals of size less than $\frac{1}{n}$.

Let $I_j$ denote any interval which appeared at the $k-j$ splitting: thus $I_k = I$, and $I_0$ is some interval such that $m(I_0) < \frac{1}{n}$. We show now by induction on $j$ that

$$P\left\{ g_{I_j} \geq \epsilon \right\} \leq c_j \, M\epsilon^{-\nu} \, m(I_j)^{1+\beta}, \tag{2.5}$$

where the sequence $c_j$ is given by

$$c_0 = 1, \; (c_j)^{\frac{1}{\nu+1}} = 1 + \left( \frac{c_{j-1}}{2^\beta} \right)^{\frac{1}{\nu+1}}. \tag{2.6}$$

For $j = 0$, this is just assumption (2.3). For $j \geq 1$, let $I_j'$ and $I_j''$ denote the two halves of $I_j$. Then

$$\begin{aligned} P\left\{ \max(g_{I_j'}, g_{I_j''}) > \epsilon \right\} &\leq 2P\left\{ g_{I_j'} > \epsilon \right\} \\ &\leq 2\, c_{j-1} M\epsilon^{-\nu} \, m(I_j')^{1+\beta} \\ &= \frac{c_{j-1}}{2^\beta} M \, \epsilon^{-\nu} \, m(I_j)^{1+\beta}. \end{aligned}$$

Relation (2.6) now follows from the fact, that if $X, Y, Z$ are random variables satisfying $0 \leq X \leq Y + Z$, $P\{Z \geq \epsilon\} \leq M\epsilon^{-\nu}$ and $P\{Y \geq \epsilon\} \leq cM\epsilon^{-\nu}$, then

$$\begin{aligned} P\{X \geq \epsilon\} &\leq \inf_{0 \leq \lambda \leq 1} \Big[ P\{Y \geq (1-\lambda)\epsilon\} + P\{Z \geq \lambda\epsilon\} \Big] \\ &\leq \inf_{0 \leq \lambda \leq 1} M\epsilon^{-\nu} \Big[ c(1-\lambda)^{-\nu} + \lambda^{-\nu} \Big] \\ &= M\epsilon^{-\nu} (1 + c^{\frac{1}{\nu+1}})^{\nu+1} \end{aligned} \tag{2.7}$$

Using (2.6) with $\beta = 0$ yields

$$(c_k)^{\frac{1}{\nu-1}} = 1 + (c_{k-1})^{\frac{1}{\nu-1}} = \cdots = k + 1.$$

and

$$c_k = (k+1)^{\nu-1} \leq (\log_2 4n)^{\nu-1}.$$

On the other hand, if $\beta > 0$, then

$$(c_k)^{\frac{1}{\nu-1}} \leq \sum_{j=0}^{\infty} 2^{-\beta j/(\nu-1)} = \left[1 - 2^{-\beta/(\nu-1)}\right]^{-1},$$

yielding (2.4). ∎

## 3. Inner and outer sub-triadditivity

We focus at first on *deterministic* functions $H$ defined on

$$[0,1]^3_{\leq} = \left\{(x_1, x_2, x_3) : 0 \leq x_1 \leq x \leq x_2 \leq 1\right\}$$

or defined on all of $\mathbf{R}^3$.

**Definition.** A function $H : [0,1]^3_{\leq} \to \mathbf{R}^+$ is called *inner sub-triadditive* if it satisfies

$$H(x_1, x, x_2) \leq H(x_1, x, y) + H(x_1, y, x_2) \tag{3.1}$$

whenever $x_1 \leq x \leq y \leq x_2$, and

$$H(x_1, x, x_2) \leq H(y, x, x_2) + H(x_1, y, x_2) \tag{3.2}$$

whenever $x_1 \leq y \leq x \leq x_2$.

A function $H : \mathbf{R}^3 \to \mathbf{R}^+$ is called *inner sub-triadditive* if it satisfies (3.1) and (3.2) for any reals $x_1, x, x_2, y$.

**Definition.** A function $H : [0,1]^3_{\leq} \to \mathbf{R}^+$ is called *outer sub-triadditive* if it satisfies

$$H(x_1, x, x_2) \leq H(x_1, x, y) + H(x, x_2, y) \tag{3.3}$$

whenever $x_1 \le x \le x_2 \le y$, and

$$H(x_1, x, x_2) \le H(y, x, x_2) + H(y, x_1, x_{}) \tag{3.4}$$

whenever $y \le x_1 \le x \le x_2$.

A function $H : \mathbf{R}^3 \to \mathbf{R}^+$ is called *outer sub-triadditive* if it satisfies (3.3) and (3.4) for any $x_1, x, x_2, y$ in $\mathbf{R}$.

$J$ and $M$ are examples of functions that are both inner and outer sub-triadditive (see Appendix). We show next why inner or outer sub-triadditivity are useful properties. But first, some notation.

If $I = [t_1, t_2]$ is an interval, let $t_I$ denote the middle point in $I$, and let $I' = [t_1, t_I]$, $I'' = [t_I, t_2]$ denote the two half intervals of $I$.

**Lemma 2.** *(a) If the function* $f : [0,1]^3_{\le} \to \mathbf{R}_+$ *is inner sub-triadditive, then*

$$\bar{f}_I \le \max\left\{\bar{f}_{I'}, \bar{f}_{I''}\right\} + f(t_1, t_I, t_2). \tag{3.5}$$

*(b) If, moreover, $f$ is outer sub-triadditive, then*

$$\bar{f}^*_I \le \max\left\{\bar{f}^*_{I'}, \bar{f}^*_{I''}\right\} + \bar{f}_{I'} + \bar{f}_{I''} + f(t_1, t_I, t_2). \tag{3.6}$$

**Proof** (a) If $t \le t_I$, then by (3.1),

$$\begin{aligned} f(t_1, t, t_2) &\le f(t_1, t, t_I) + f(t_1, t_I, t_2) \\ &\le \bar{f}_{I'} + f(t_1, t_I, t_2), \end{aligned}$$

while if $t \ge t_I$, by (3.2) we have similarly

$$f(t_1, t, t_2) \le \bar{f}_{I''} + f(t_1, t_I, t_2).$$

(b) If $t_1 < a < b < t_I < c < t_2$, then

$$\begin{aligned} f(a,b,c) &\le f(a,b,t_I) + f(a,t_I,c) \text{ (by inner sub-triadditivity)} \\ &\le f(a,b,t_I) + f(t_1,t_I,c) + f(t_1,a,t_I) \text{ (by outer sub-triadditivity)} \\ &\le f(a,b,t_I) + f(t_1,a,t_I) + f(t_1,t_I,t_2) + f(t_I,c,t_2) \text{ (by outer sub-triadditivity)} \\ &\le \bar{f}^*_{I'} + \bar{f}_{I'} + \bar{f}_{I''} + f(t_1,t_I,t_2) \\ &\le R.H.S. \text{ of } (3.6). \end{aligned}$$

In the same way, we get

$$f(a,b,c) \leq R.H.S.\ of\ (3.6) \tag{3.7}$$

when $t_1 < a < t_I < b < c < t_2$, and since (3.7) is obvious when $a,b,c \in I'$ or $a,b,c \in I''$, (3.6) holds. ∎

We consider now an a.s. *random* sub-triadditive function of a process $Z$, denoted $f_Z$. Recall that we assume that the process $Z$ satisfies assumption (B). We will always assume that random sub-triadditive functions $f_Z$ also satisfy

$$f_Z(t_1,t,t_2) = f_Z(t_1',t',t_2') \tag{3.8}$$

whenever $(t_1,t_1')$, $(t,t')$ and $(t_2,t_2')$ are pairs of points for which $Z$ is constant on the interval between them.

Note that both $H_Z$ and $H_Z^*$ satisfy (3.8), where $H$ is either $J$ or $M$.

We will now prove Theorem 2.

**Proof of Theorem 2.** (a) The result follows from Proposition 1 of Section 2, applied to the functions $g_I = \bar{f}_Z(t_1,t_2)$ and $h_I = f_Z(t_1,t_I,t_2)$. Indeed, (2.1) holds by Lemma 2a. For (2.3), note that if $J = [t_1,t_2]$ is such that $t_2 - t_1 < \frac{1}{n}$, then by Assumption (B) on $Z$ and by Assumption (3.8) on $f_Z$ we have

$$\begin{aligned}
P\left\{g_J \geq \epsilon\right\} &= P\left\{\bar{f}_Z(t_1,t_2) \geq \epsilon\right\} \\
&= P\left\{\max[f_Z(t_1,t_1,t_2), f_Z(t_1,t_2,t_2)] \geq \epsilon\right\} \\
&\leq P\left\{f_Z(t_1,t_1,t_2) \geq \epsilon\right\} + P\left\{f_Z(t_1,t_2,t_2) \geq \epsilon\right\} \\
&\leq 2L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}.
\end{aligned}$$

Thus (2.3) holds with $M = 2L$, and clearly (2.2) holds also with $M = 2L$. We get then by Proposition 1

$$P\left\{\bar{f}_Z(I) \geq \epsilon\right\} \leq 2K(\nu,\beta,n)L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta},$$

with $K(\nu,\beta,n)$ given by (2.4).

(b) The result follows again from Proposition 1, this time applied to

$$g_I = \bar{f}^{*}_{Z}(t_1, t_2)$$

and

$$h_I = \bar{f}_Z(t_1, t_I) + \bar{f}_Z(t_I, t_2) + f_Z(t_1, t_I, t_2).$$

Relation (2.1) holds by Lemma 2b.

We check now (2.2).

$$\begin{aligned} P\left\{h_I \geq \epsilon\right\} &\leq \inf_{0 \leq \lambda \leq 1}\Big[P\left\{\bar{f}_Z(I') > (\tfrac{1-\lambda}{2})\epsilon\right\} + P\left\{f_Z(I'') \geq (\tfrac{1-\lambda}{2})\epsilon\right\} \\ &\quad + L(\lambda\epsilon)^{-\nu}(t_2 - t_1)^{1+\beta}\Big] \\ &\leq \inf_{0 \leq \lambda \leq 1}\Big[c(1-\lambda)^{-\nu} + \lambda^{-\nu}\Big] L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}. \end{aligned}$$

The last step holds by part (a), with

$$c = 2^{1+1-\nu-1-\beta}K(\nu, \beta, n) = 2^{1-\nu-\beta}K(\nu, \beta, n).$$

As in (2.7), we then get

$$\begin{aligned} P\left\{h_I \geq \epsilon\right\} &\leq \left(1 + 2^{1-\beta/(\nu+1)}K(\nu, \beta, n)^{\frac{1}{\nu+1}}\right)^{\nu+1} L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta} \\ &\leq K(\nu, \beta, n)\left[1 + 2^{\ \beta/(\nu+1)}\right]^{\nu+1} L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}, \end{aligned} \tag{3.9}$$

since $K(\nu, \beta, n) \geq 1$.

To check (2.3), let $J = [t_1, t_2]$ be such that $t_2 - t_1 \leq \frac{1}{n}$. Then, as in part (a),

$$P\left\{\bar{f}^{*}_{J} \geq \epsilon\right\} \leq 2L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}.$$

Hence (2.2) and (2.3) hold with

$$M = K(\nu, \beta, n)\max\left\{2, [1 + 2^{1\ \ \beta/(\nu+1)}]^{\nu+1}\right\}.$$

Applying Proposition 1, we get

$$P\left\{\bar{f}^{*}_{Z}(I) \geq \epsilon\right\} \leq A(\nu, \beta, n)L\epsilon^{-\nu}(t_2 - t_1)^{1+\beta}$$

where

$$(3.10) \qquad A(\nu,\beta,n) = K^2(\nu,\beta,n)\max\left\{2, [1+2^{-\beta/(\nu-1)}]^{\nu+1}\right\}$$

$$= \begin{cases} [1-2^{-\beta/(\nu+1)}]^{-(2\nu+2)}\max\{2, [1+2^{-\beta/(\nu+1)}]^{\nu+1}\} & \text{if } \beta > 0 \\ 3^{\nu+1}(\log_2(4n))^{2\nu+2} & \text{if } \beta = 0. \end{cases}$$

This concludes the proof. ∎

## Appendix

**Lemma A1**. *The deterministic functions $J$ and $M$ defined in (1.1) are inner and outer sub-triadditive.*

**Proof**. (a) *Inner sub-triadditivity*

Since $J$ and $M$ are symmetric in $x_1, x_2$, it is enough to check (3.1). For $J$, we must show that

$$(A.1) \qquad |x-x_1| \wedge |x-x_2| \le |x-x_1| \wedge |x-y| + |x_1-y| \wedge |y-x_2|$$

holds.

If $|x-x_1| < |x-y|$, then

$$\text{R.H.S. of (A.1)} \ \ge |x-x_1| \ge \ \text{L.H.S. of (A.1)}.$$

If $|x-y| < |x-x_1|$, then either the R.H.S. of (A.1) equals $|x-y|+|x_1-y| \ge |x-x_1|$, or the R.H.S. of (A.1) equals $|x-y|+|y-x_2| \ge |x-x_2|$. Hence (A.1) holds.

Now for $M$, we check again (3.1) in different cases:

$$(a) \qquad x \in [x_1, x_2]; \text{ then } M(x_1,x,x_2) = 0.$$

If $x \notin [x_1, x_2]$, w.l.o.g., let $x < x_1 < x_2$; we have the following subcases:

$$(b) \qquad y < x < x_1 < x_2; \quad \text{then}$$

$$M(x_1,x,x_2) = x_1 - x \le x_1 - y = M(x_1,y,x_2).$$

(c) $x < y < x_1 < x_2$: then

$$M(x_1, x, x_2) = x_1 - x = (y - x) + (x_1 - y) = M(x_1, x, y) + M(x_1, y, x_2).$$

(d) $x < x_1 < y$: then

$$M(x_1, x, x_2) = x_1 - x = M(x_1, x, y).$$

(b) *Outer sub-triadditivity*

Since $J, M$ are symmetric in $x_1, x_2$, it is enough to check (3.3). For $J$, we must show that

$$(A.2) \qquad |x - x_1| \wedge |x - x_2| \leq |x - x_1| \wedge |x - y| + |x_2 - x| \wedge |x_2 - y|$$

holds. The only case different from part (a) is when $|x - y| < |x - x_1|$ and $|x_2 - x| < |x_2 - y|$. In this case,

$$\text{R.H.S. of (A.2)} \quad = |x - y| + |x_2 - x| > |x_2 - y| > |x_2 - x| > \quad \text{L.H.S. of (A.2)}.$$

For $H = M$, we assume, w.l.o.g. $x < x_1 < x_2$. If $y < x < x_1 < x_2$,

$$M(x_1, x, x_2) = x_1 - x \leq x_2 - x = M(x, x_2, y).$$

If $x < y < x_1 < x_2$,

$$M(x_1, x, x_2) = x_1 - x \leq (y - x) + (x_2 - y) = M(x_1, x, y) + M(x, x_2, y).$$

If $x < x_1 < y$,

$$M(x_1, x, x_2) = x_1 - x_2 = M(x_1, x, y). \qquad \blacksquare$$

**Lemma A2.** *If $H : \mathbf{R}^3 \to \mathbf{R}^+$ is inner sub-triadditive, then the function $H_Z^* : [0,1]^3_{\leq} \to \mathbf{R}^+$ defined by*

$$H_Z^*(t_1, t, t_2) = \sup_{\substack{t_1' \in [t_1, t] \\ t_2' \in [t, t_2]}} H(Z(t_1'), Z(t), Z(t_2'))$$

*is inner sub-triadditive.*

**Proof**. Since $H_Z^*(t_1, t, t_2)$ is defined only for ordered triples $t_1 < t < t_2$, we have to check that

(A.3a) If $t_1 < t < u < t_2$, then $H_Z^*(t_1, t, t_2) \leq H_Z^*(t_1, t, u) + H_Z^*(t_1, u, t_2)$

and

(A.3b) If $t_1 < u < t < t_2$, then $H_Z^*(t_1, t, t_2) \leq H_Z^*(u, t, t_2) + H_Z^*(t_1, u, t_2)$.

Since the proofs are similar, we show only (A.3a). Also, for convenience, assume that the sup is obtained, that is,

$$H_Z^*(t_1, t, t_2) = H(Z(t_1'). Z(t). Z(t_2')),$$

for some $t_1', t_2'$, with $t_1 \leq t_1' \leq t \leq t_2' \leq t_2$.

Let now $t \leq u \leq t_2$; if $u \geq t_2'$, then, obviously, $H_Z^*(t_1, t, t_2) = H_Z^*(t_1, t, u)$. and (A.3a) holds. Suppose, hence $u \leq t_2'$; by the inner sub-triadditivity of $H$,

$$\begin{aligned} H(Z(t_1'), Z(t), Z(t_2')) &\leq H(Z(t_1'), Z(t), Z(u)) + H(Z(t_1'), Z(u), Z(t_2')) \\ &\leq H_Z^*(t_1, t, u) + H_Z^*(t_1, u, t_2). \end{aligned}$$

The result follows by taking sup in the L.H.S. ∎

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